

Algebraic aspects of the correlation functions of the integrable higher-spin XXZ spin chains with arbitrary entries

Tetsuo Deguchi*

Department of Physics, Graduate School of Humanities and Sciences, Ochanomizu University,

2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan

**E-mail: deguchi@phys.ocha.ac.jp*

Chihiro Matsui^{† ‡}

*Department of Physics, Graduate School of Science, the University of Tokyo
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*

[†] *CREST, JST, 4-1-8 Honcho Kawaguchi, Saitama, 332-0012, Japan*

[‡] *E-mail: matsui@spin.phys.s.u-tokyo.ac.jp*

We discuss some fundamental properties of the XXZ spin chain, which are important in the algebraic Bethe-ansatz derivation for the multiple-integral representations of the spin- s XXZ correlation function with an arbitrary product of elementary matrices.¹ For instance, we construct Hermitian conjugate vectors in the massless regime and introduce the spin- s Hermitian elementary matrices.

Keywords: Correlation functions; XXZ spin chains; algebraic Bethe ansatz; quantum groups; multiple-integral representations.

1. Introduction

The correlation functions of the spin-1/2 XXZ spin chain have attracted much interest in mathematical physics through the last two decades. One of the most fundamental results is the exact derivation of their multiple-integral representations. The multiple-integral representations of the XXZ correlation functions were derived for the first time by making use of the q -vertex operators through the affine quantum-group symmetry in the massive regime for the infinite lattice at zero temperature.^{2,3} They were also derived in the massless regime by solving the q -KZ equations.^{4,5} Making use of algebraic Bethe-ansatz techniques such as scalar products,^{6–10} the multiple-integral representations were derived for the XXZ correlation func-

tions under a non-zero magnetic field.¹¹ They were extended into those at finite temperatures,¹² and even for a large finite chain.¹³ Interestingly, they are factorized in terms of single integrals.¹⁴ Furthermore, the asymptotic expansion of a correlation function of the XXZ model has been systematically discussed.¹⁵ Thus, the exact study of the XXZ correlation functions should play an important role not only in the mathematical physics of integrable models but also in many areas of theoretical physics.

The Hamiltonian of the spin-1/2 XXZ spin chain under the periodic boundary conditions is given by

$$\mathcal{H}_{\text{XXZ}} = \frac{1}{2} \sum_{j=1}^L (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z) . \quad (1.1)$$

Here σ_j^a ($a = X, Y, Z$) are the Pauli matrices defined on the j th site and Δ denotes the XXZ coupling. We define parameter q by

$$\Delta = (q + q^{-1})/2 . \quad (1.2)$$

We define η by $q = \exp \eta$. In the massive regime: $\Delta > 1$, we put $\eta = \zeta$ with $\zeta > 0$. At $\Delta = 1$ (i.e. $q = 1$) the Hamiltonian (1.1) gives the antiferromagnetic Heisenberg (XXX) chain. In the massless regime: $-1 < \Delta \leq 1$, we set $\eta = i\zeta$, and we have $\Delta = \cos \zeta$ with $0 \leq \zeta < \pi$ for the spin-1/2 XXZ spin chain (1.1). In the paper we consider a massless region: $0 \leq \zeta < \pi/2s$ for the ground-state of the integrable spin- s XXZ spin chain.

Recently, the correlation functions and form factors of the integrable higher-spin XXX and XXZ spin chains have been derived by the algebraic Bethe-ansatz method.^{1,16–18} The solvable higher-spin generalizations of the XXX and XXZ spin chains have been derived by the fusion method in several references.^{19–25} In the region: $0 \leq \zeta < \pi/2s$, the spin- s ground-state should be given by a set of string solutions.^{26,27} Furthermore, the critical behavior should be given by the SU(2) WZWN model of level $k = 2s$ with central charge $c = 3s/(s+1)$.^{24,28–40} For the integrable higher-spin XXZ spin chain correlation functions have been discussed in the massive regime by the method of q -vertex operators.^{41–44}

In the present paper we discuss several important points in the algebraic Bethe-ansatz derivation of the correlation functions for the integrable spin- s XXZ spin chain where s is an arbitrary integer or a half-integer.¹ In particular, we briefly discuss a rigorous derivation of the finite-sum expression of correlation functions for the spin- s XXZ spin chain.

The content of the paper consists of the following. In section 2 we formulate the R -matrices in the homogeneous and principal gradings, respectively.

They are related to each other by a similarity transformation. In section 3 we introduce the Hermitian elementary matrices and construct conjugate basis vectors for the spin- s Hilbert space in the massless regime. In section 4 we construct fusion monodromy matrices. In section 5, we first present formulas¹ for expressing the Hermitian elementary matrices in terms of global operators. Then, we review the multiple-integral representations of the spin- s XXZ correlation function for an arbitrary product of elementary matrices.¹ In section 6 we briefly sketch the derivation of the finite-sum expression of correlation functions for the spin- s XXZ spin chain, which leads to the multiple-integral representation in the thermodynamic limit. Here the spin-1/2 case corresponds to eq. (5.6) of Ref. 11.

2. Symmetric and asymmetric R -matrices

2.1. R -matrix and the monodromy matrix of type $(1, 1^{\otimes L})$

Let us now define the R -matrix of the XXZ spin chain.^{7-9,11} For two-dimensional vector spaces V_1 and V_2 , we define $R^\pm(\lambda_1 - \lambda_2)$ acting on $V_1 \otimes V_2$ by

$$R^\pm(\lambda_1 - \lambda_2) = \sum_{a,b,c,d=0,1} R^\pm(u)_{cd}^{ab} e^{a,c} \otimes e^{b,d} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^\mp(u) & 0 \\ 0 & c^\pm(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.1)$$

where $u = \lambda_1 - \lambda_2$, $b(u) = \sinh u / \sinh(u + \eta)$ and $c^\pm(u) = \exp(\pm u) \sinh \eta / \sinh(u + \eta)$. We denote by $e^{a,b}$ a unit matrix that has only one nonzero element equal to 1 at entry (a, b) where $a, b = 0, 1$.

The asymmetric R -matrix (2.1), $R^\pm(u)$, is compatible with the homogeneous grading of $U_q(\widehat{sl}_2)$.^{3,18} We denote by $R^{(p)}(u)$ or simply by $R(u)$ the symmetric R -matrix where $c^\pm(u)$ of (2.1) are replaced by $c(u) = \sinh \eta / \sinh(u + \eta)$.¹⁸ It is compatible with the affine quantum group $U_q(\widehat{sl}_2)$ of the principal grading.^{3,18} Hereafter, we denote them concisely by $R^{(w)}(u)$ with $w = \pm$ and p , where $w = +$ and $w = p$ in superscript show the homogeneous and the principal grading, respectively.

Let s be an integer or a half-integer. We shall mainly consider the tensor product $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$ of $(2s + 1)$ -dimensional vector spaces $V_j^{(2s)}$ with parameters ξ_j , where $L = 2sN_s$. Here N_s denotes the lattice size of the spin- s chain. In general, we may consider the tensor product $V_0^{(2s_0)} \otimes V_1^{(2s_1)} \otimes \cdots \otimes V_r^{(2s_r)}$ with $2s_1 + \cdots + 2s_r = L$, where $V_j^{(2s_j)}$ have parameters λ_j or ξ_j for $j = 1, 2, \dots, r$. For a given set of matrix elements $A_{b,\beta}^{a,\alpha}$ for

$a, b = 0, 1, \dots, 2s_j$ and $\alpha, \beta = 0, 1, \dots, 2s_k$, we define operator $A_{j,k}$ by

$$A_{j,k} = \sum_{a,b=1}^{\ell} \sum_{\alpha,\beta} A_{b,\beta}^{a,\alpha} I_0^{(2s_0)} \otimes I_1^{(2s_1)} \otimes \dots \otimes I_{j-1}^{(2s_{j-1})} \otimes E_j^{a,b(2s_j)} \otimes I_{j+1}^{(2s_{j+1})} \otimes \dots \otimes I_{k-1}^{(2s_{k-1})} \otimes E_k^{\alpha,\beta(2s_k)} \otimes I_{k+1}^{(2s_{k+1})} \otimes \dots \otimes I_r^{(2s_r)}. \quad (2.2)$$

Here $E_j^{a,b(2s_j)}$ denote the elementary matrices in the spin- s_j representation, each of which has nonzero matrix element only at entry (a, b) .

When $s_0 = \ell/2$ and $s_1 = \dots = s_r = s$, we denote the type by $(\ell, (2s)^{\otimes N_s})$. In particular, for $s = 1/2$, we denote it by $(\ell, 1^{\otimes L})$.

2.2. Gauge transformations

Let us introduce operators Φ_j with arbitrary parameters ϕ_j for $j = 0, 1, \dots, L$ as follows:

$$\Phi_j = \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi_j} \end{pmatrix}_{[j]} = I^{\otimes(j)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{\phi_j} \end{pmatrix} \otimes I^{\otimes(L-j)}. \quad (2.3)$$

In terms of $\chi_{jk} = \Phi_j \Phi_k$, we define a similarity transformation on the R -matrix by

$$R_{jk}^{\chi} = \chi_{jk} R_{jk} \chi_{jk}^{-1}. \quad (2.4)$$

Explicitly, the following two matrix elements are transformed.

$$\left(R_{jk}^{\chi}\right)_{12}^{21} = c(\lambda_j, \lambda_k) e^{\phi_j - \phi_k}, \quad \left(R_{jk}^{\chi}\right)_{21}^{12} = c(\lambda_j, \lambda_k) e^{-\phi_j + \phi_k}. \quad (2.5)$$

Putting $\phi_j = \lambda_j$ for $j = 0, 1, \dots, L$ in eq. (2.3) we have

$$R_{jk}^{\pm}(\lambda_j, \lambda_k) = (\chi_{jk})^{\pm 1} R_{jk}(\lambda_j, \lambda_k) (\chi_{jk})^{\mp 1} \quad (j, k = 0, 1, \dots, L). \quad (2.6)$$

Thus, the asymmetric R -matrices $R_{12}^{\pm}(\lambda_1, \lambda_2)$ are derived from the symmetric one through the gauge transformation χ_{jk} .

2.3. Monodromy matrices

Applying definition (2.2) for matrix elements $R(u)_{cd}^{ab}$ of a given R -matrix, $R^{(w)}(u)$ for $w = \pm$ and p , we define R -matrices $R_{jk}^{(w)}(\lambda_j, \lambda_k) = R_{jk}^{(w)}(\lambda_j - \lambda_k)$ for integers j and k with $0 \leq j < k \leq L$. For integers j, k and ℓ with $0 \leq j < k < \ell \leq L$, the R -matrices satisfy the Yang-Baxter equations

$$\begin{aligned} & R_{jk}^{(w)}(\lambda_j - \lambda_k) R_{j\ell}^{(w)}(\lambda_j - \lambda_\ell) R_{k\ell}^{(w)}(\lambda_k - \lambda_\ell) \\ &= R_{k\ell}^{(w)}(\lambda_k - \lambda_\ell) R_{j\ell}^{(w)}(\lambda_j - \lambda_\ell) R_{jk}^{(w)}(\lambda_j - \lambda_k). \end{aligned} \quad (2.7)$$

Let us introduce notation for expressing products of R -matrices.

$$\begin{aligned} R_{1,23\dots n}^{(w)} &= R_{1n}^{(w)} \cdots R_{13}^{(w)} R_{12}^{(w)}, \\ R_{12\dots n-1,n}^{(w)} &= R_{1n}^{(w)} R_{2n}^{(w)} \cdots R_{n-1,n}^{(w)}. \end{aligned} \quad (2.8)$$

Here $R_{ab}^{(w)}$ denote the R -matrix $R_{ab}^{(w)} = R_{ab}^{(w)}(\lambda_a - \lambda_b)$ for $a, b = 1, 2, \dots, n$.

We now define the monodromy matrix of type $(1, 1^{\otimes L} w)$, i.e. of type $(1, 1^{\otimes L})$ with grading w . Expressing the symbol $(1, 1^{\otimes L})$ briefly as $(1, 1)$ in superscript we define it by

$$\begin{aligned} T_{0,12\dots L}^{(1,1^w)}(\lambda_0; \{w_j\}_L) &= R_{0L}^+(\lambda_0 - w_L) \cdots R_{02}^+(\lambda_0 - w_2) R_{01}^+(\lambda_0 - w_1) \\ &= R_{0L}^{(w)} R_{0L-1}^{(w)} \cdots R_{01}^{(w)} = R_{0,12\dots L}^{(w)}(\lambda_0; \{w_j\}_L). \end{aligned} \quad (2.9)$$

Here we have put $\lambda_j = w_j$ for $j = 1, 2, \dots, L$. They are arbitrary. We call them *inhomogeneous parameters*. We express the operator-valued matrix elements of the monodromy matrix as follows.

$$T_{0,12\dots L}^{(1,1^+)}(\lambda; \{w_j\}_L) = \begin{pmatrix} A_{12\dots L}^{(1+)}(\lambda; \{w_j\}_L) & B_{12\dots L}^{(1+)}(\lambda; \{w_j\}_L) \\ C_{12\dots L}^{(1+)}(\lambda; \{w_j\}_L) & D_{12\dots L}^{(1+)}(\lambda; \{w_j\}_L) \end{pmatrix}. \quad (2.10)$$

We also denote the operator-valued matrix elements by $[T_{0,12\dots L}^{(1,1^+)}(\lambda; \{w_j\}_L)]_{a,b}$ for $a, b = 0, 1$. Here $\{w_j\}_L$ denotes the inhomogeneous parameters w_1, w_2, \dots, w_L . Hereafter we denote by $\{\mu_j\}_N$ the set of N numbers or parameters μ_1, \dots, μ_N .

The monodromy matrix of principal grading, $T_{0,12\dots L}^{(1,1^p)}(\lambda; \{w_j\}_L)$, is related to that of homogeneous grading via similarity transformation $\chi_{01\dots L} = \Phi_0 \Phi_1 \cdots \Phi_L$ as follows.¹⁸

$$\begin{aligned} T_{0,12\dots L}^{(1,1^+)}(\lambda; \{w_j\}_L) &= \chi_{012\dots L} T_{0,12\dots L}^{(1,1^p)}(\lambda; \{w_j\}_L) \chi_{012\dots L}^{-1} \\ &= \begin{pmatrix} \chi_{12\dots L} A_{12\dots L}^{(1^p)}(\lambda; \{w_j\}_L) \chi_{12\dots L}^{-1} & e^{-\lambda_0} \chi_{12\dots L} B_{12\dots L}^{(1^p)}(\lambda; \{w_j\}_L) \chi_{12\dots L}^{-1} \\ e^{\lambda_0} \chi_{12\dots L} C_{12\dots L}^{(1^p)}(\lambda; \{w_j\}_L) \chi_{12\dots L}^{-1} & \chi_{12\dots L} D_{12\dots L}^{(1^p)}(\lambda; \{w_j\}_L) \chi_{12\dots L}^{-1} \end{pmatrix}. \end{aligned}$$

In Ref.¹⁸ operator $A^{(1^+)}(\lambda)$ has been written as $A^+(\lambda)$.

2.4. Operator \check{R} : Another form of the R -matrix

Let V_1 and V_2 be $(2s+1)$ -dimensional vector spaces. We define permutation operator $\Pi_{1,2}$ by

$$\Pi_{1,2} v_1 \otimes v_2 = v_2 \otimes v_1, \quad v_1 \in V_1, v_2 \in V_2. \quad (2.11)$$

In the spin-1/2 case, we define operator $\check{R}_{j,j+1}^{(w)}(u)$ by

$$\check{R}_{j,j+1}^{(w)}(u) = \Pi_{j,j+1} R_{j,j+1}^{(w)}(u). \quad (2.12)$$

3. The quantum group invariance

3.1. Quantum group $U_q(sl_2)$

The quantum algebra $U_q(sl_2)$ is an associative algebra over \mathbf{C} generated by X^\pm, K^\pm with the following relations:^{45–47}

$$\begin{aligned} KK^{-1} &= KK^{-1} = 1, \quad KX^\pm K^{-1} = q^{\pm 2}X^\pm, \\ [X^+, X^-] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned} \quad (3.1)$$

The algebra $U_q(sl_2)$ is also a Hopf algebra over \mathbf{C} with comultiplication

$$\begin{aligned} \Delta(X^+) &= X^+ \otimes 1 + K \otimes X^+, \quad \Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^-, \\ \Delta(K) &= K \otimes K, \end{aligned} \quad (3.2)$$

and antipode: $S(K) = K^{-1}, S(X^+) = -K^{-1}X^+, S(X^-) = -X^-K$, and coproduct: $\epsilon(X^\pm) = 0$ and $\epsilon(K) = 1$.

It is easy to see that the asymmetric R -matrix gives an intertwiner of the spin-1/2 representation of $U_q(sl_2)$:

$$R_{12}^+(u)\Delta(x) = \tau \circ \Delta(x)R_{12}^+(u) \quad \text{for } x = X^\pm, K. \quad (3.3)$$

Here we remark that spectral parameter u is arbitrary and independent of X^\pm or K .

3.2. Temperley-Lieb algebra

Operators $\check{R}_{j,j+1}^\pm(u)$ are decomposed in terms of the generators of the Temperley-Lieb algebra as follows.[?]

$$\check{R}_{j,j+1}^\pm(u) = I - b(u)U_j^\pm. \quad (3.4)$$

U_j^\pm s (U_j^\pm s) satisfy the defining relations of the Temperley-Lieb algebra:[?]

$$\begin{aligned} U_j^\pm U_{j+1}^\pm U_j^\pm &= U_j^\pm, \\ U_{j+1}^\pm U_j^\pm U_{j+1}^\pm &= U_j^\pm, \quad \text{for } j = 0, 1, \dots, L-2, \\ (U_j^\pm)^2 &= (q + q^{-1})U_j^\pm \quad \text{for } j = 0, 1, \dots, L-1, \\ U_j^\pm U_k^\pm &= U_k^\pm U_j^\pm \quad \text{for } |j - k| > 1. \end{aligned} \quad (3.5)$$

We remark that the asymmetric R -matrices $\check{R}_{j,j+1}^\pm(u)$ derived from the symmetric R -matrix through the gauge transformation are related to the Jones polynomial.⁴⁹

3.3. Basis vectors of spin- $\ell/2$ representation of $U_q(sl_2)$

Let us introduce the q -integer for an integer n by $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. We define the q -factorial $[n]_q!$ for integers n by

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q. \quad (3.6)$$

For integers m and n satisfying $m \geq n \geq 0$ we define the q -binomial coefficients as follows

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q! [n]_q!}. \quad (3.7)$$

We now define the basis vectors of the $(\ell+1)$ -dimensional irreducible representation of $U_q(sl_2)$, $|\ell, n\rangle$ for $n = 0, 1, \dots, \ell$ as follows. We define $|\ell, 0\rangle$ by

$$|\ell, 0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_\ell. \quad (3.8)$$

Here $|\alpha\rangle_j$ for $\alpha = 0, 1$ denote the basis vectors of the spin-1/2 representation defined on the j th position in the tensor product. We define $|\ell, n\rangle$ for $n \geq 1$ and evaluate them as follows¹⁸.

$$\begin{aligned} |\ell, n\rangle &= \left(\Delta^{(\ell-1)}(X^-) \right)^n |\ell, 0\rangle \frac{1}{[n]_q!} \\ &= \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- |0\rangle q^{i_1 + i_2 + \cdots + i_n - n\ell + n(n-1)/2}. \end{aligned} \quad (3.9)$$

We define the conjugate vectors explicitly by the following:

$$\langle \ell, n| = \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} q^{n(\ell-n)} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \langle 0| \sigma_{i_1}^+ \cdots \sigma_{i_n}^+ q^{i_1 + \cdots + i_n - n\ell + n(n-1)/2}. \quad (3.10)$$

It is easy to show the normalization conditions:¹⁸ $\langle \ell, n| |\ell, n\rangle = 1$. In the massive regime where $q = \exp \eta$ with real η , conjugate vectors $\langle \ell, n|$ are Hermitian conjugate to vectors $|\ell, n\rangle$.

3.4. Conjugate vectors

In order to construct Hermitian elementary matrices in the massless regime where $|q| = 1$, we now introduce another set of dual basis vectors. For a given nonzero integer ℓ we define $\widetilde{\langle \ell, n|}$ for $n = 0, 1, \dots, \ell$, by

$$\widetilde{\langle \ell, n|} = \begin{pmatrix} \ell \\ n \end{pmatrix}^{-1} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \langle 0| \sigma_{i_1}^+ \cdots \sigma_{i_n}^+ q^{-(i_1 + \cdots + i_n) + n\ell - n(n-1)/2}. \quad (3.11)$$

They are conjugate to $|\ell, n\rangle$: $\langle \ell, m | |\ell, n\rangle = \delta_{m,n}$. Here we have denoted the binomial coefficients for integers ℓ and n with $0 \leq n \leq \ell$ as follows.

$$\binom{\ell}{n} = \frac{\ell!}{(n!)n!}. \quad (3.12)$$

We now introduce vectors $|\ell, n\rangle$ which are Hermitian conjugate to $\langle \ell, n|$ when $|q| = 1$ for positive integers ℓ with $n = 0, 1, \dots, \ell$. Setting the norm of $|\ell, n\rangle$ such that $\langle \ell, n | |\ell, n\rangle = 1$, vectors $|\ell, n\rangle$ are given by

$$\sum_{1 \leq i_1 < \dots < i_n \leq \ell} \sigma_{i_1}^- \dots \sigma_{i_n}^- |0\rangle q^{-(i_1 + \dots + i_n) + n\ell - n(n-1)/2} \begin{bmatrix} \ell \\ n \end{bmatrix}_q q^{-n(\ell-n)} \binom{\ell}{n}^{-1}. \quad (3.13)$$

We have the following normalization condition:

$$\langle \ell, n | |\ell, n\rangle = \begin{bmatrix} \ell \\ n \end{bmatrix}_q^2 \binom{\ell}{n}^{-2}. \quad (3.14)$$

3.5. Hermitian elementary matrices

In the massless regime we define elementary matrices $\tilde{E}^{m,n(2s+)}$ for $m, n = 0, 1, \dots, 2s$ by

$$\tilde{E}^{m,n(2s+)} = |\widetilde{2s, m}\rangle \langle \widetilde{2s, n}|. \quad (3.15)$$

In the massless regime where $|q| = 1$, matrix $|\ell, n\rangle \langle \ell, n|$ is Hermitian: $(|\ell, n\rangle \langle \ell, n|)^\dagger = |\ell, n\rangle \langle \ell, n|$. However, in order to define projection operators \tilde{P} such that $P\tilde{P} = P$, we have formulated vectors $|\ell, n\rangle$.

3.6. Projection operators

We define projection operators acting on the 1st to the ℓ th tensor-product spaces by

$$P_{12\dots\ell}^{(\ell)} = \sum_{n=0}^{\ell} |\ell, n\rangle \langle \ell, n|. \quad (3.16)$$

Let us now introduce another set of projection operators $\tilde{P}_{1\dots\ell}^{(\ell)}$ as follows.

$$\tilde{P}_{1\dots\ell}^{(\ell)} = \sum_{n=0}^{\ell} |\widetilde{\ell, n}\rangle \langle \ell, n|. \quad (3.17)$$

Projector $\tilde{P}_{1\dots\ell}^{(\ell)}$ is idempotent: $(\tilde{P}_{1\dots\ell}^{(\ell)})^2 = \tilde{P}_{1\dots\ell}^{(\ell)}$. In the massless regime where $|q| = 1$, it is Hermitian: $(\tilde{P}_{1\dots\ell}^{(\ell)})^\dagger = \tilde{P}_{1\dots\ell}^{(\ell)}$. From (3.16) and (3.17), we show the following properties:

$$P_{12\dots\ell}^{(\ell)} \tilde{P}_{1\dots\ell}^{(\ell)} = P_{12\dots\ell}^{(\ell)}, \quad (3.18)$$

$$\tilde{P}_{1\dots\ell}^{(\ell)} P_{12\dots\ell}^{(\ell)} = \tilde{P}_{1\dots\ell}^{(\ell)}. \quad (3.19)$$

In the tensor product of quantum spaces, $V_1^{(2s)} \otimes \dots \otimes V_{N_s}^{(2s)}$, we define $\tilde{P}_{12\dots L}^{(2s)}$ by

$$\tilde{P}_{12\dots L}^{(2s)} = \prod_{i=1}^{N_s} \tilde{P}_{2s(i-1)+1}^{(2s)}. \quad (3.20)$$

Here we recall $L = 2sN_s$.

The projection operators are also constructed by the fusion method. For $\ell > 2$ we can construct projection operators inductively with respect to ℓ as follows.^{25,46}

$$P_{12\dots\ell}^{(\ell)} = P_{12\dots\ell-1}^{(\ell-1)} \tilde{R}_{\ell-1,\ell}^+((\ell-1)\eta) P_{12\dots\ell-1}^{(\ell-1)}. \quad (3.21)$$

The projection operator $P_{12\dots\ell}^{(\ell)}$ gives a q -analogue of the full symmetrizer of the Young operators for the Hecke algebra.⁴⁶

4. Fusion construction

4.1. Higher-spin monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$

We now set the inhomogeneous parameters w_j for $j = 1, 2, \dots, L$, as N_s sets of complete $2s$ -strings.¹⁸ We define $w_{(b-1)\ell+\beta}^{(2s)}$ for $\beta = 1, \dots, 2s$, as follows.

$$w_{2s(b-1)+\beta}^{(2s)} = \xi_b - (\beta - 1)\eta, \quad \text{for } b = 1, 2, \dots, N_s. \quad (4.1)$$

We shall define the monodromy matrix of type $(1, (2s)^{\otimes N_s})$ associated with homogeneous grading. We first define the massless monodromy matrix by

$$\begin{aligned} \tilde{T}_{0,12\dots N_s}^{(1,2s+)}(\lambda_0; \{\xi_b\}_{N_s}) &= \tilde{P}_{12\dots L}^{(2s)} R_{0,1\dots L}^{(1,1+)}(\lambda_0; \{w_j^{(2s)}\}_L) \tilde{P}_{12\dots L}^{(2s)} \\ &= \left(\tilde{A}^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \tilde{B}^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \right) \\ &\quad \left(\tilde{C}^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \tilde{D}^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \right). \end{aligned} \quad (4.2)$$

Let us introduce a set of $2s$ -strings with small deviations from the set of complete $2s$ -strings.

$$\begin{aligned} w_{2s(b-1)+\beta}^{(2s;\epsilon)} &= \xi_b - (\beta - 1)\eta + \epsilon r_b^{(\beta)}, \quad \text{for } b = 1, 2, \dots, N_s, \\ &\text{and } \beta = 1, 2, \dots, 2s. \end{aligned} \quad (4.3)$$

Here ϵ is very small and $r_b^{(\beta)}$ are generic parameters. We express the elements of the monodromy matrix $T^{(1,1)}$ with inhomogeneous parameters given by $w_j^{(2s;\epsilon)}$ for $j = 1, 2, \dots, L$ as follows.

$$T_{0,12\dots L}^{(1,1+)}(\lambda; \{w_j^{(2s;\epsilon)}\}_L) = \begin{pmatrix} A_{12\dots L}^{(2s+;\epsilon)}(\lambda) & B_{12\dots L}^{(2s+;\epsilon)}(\lambda) \\ C_{12\dots L}^{(2s+;\epsilon)}(\lambda) & D_{12\dots L}^{(2s+;\epsilon)}(\lambda) \end{pmatrix}. \quad (4.4)$$

Here $A_{12\dots L}^{(2s+;\epsilon)}(\lambda)$ denotes $A_{12\dots L}^{(1+)}(\lambda; \{w_j^{(2s;\epsilon)}\}_L)$.

$$\tilde{A}_{12\dots N_s}^{(2s+)}(\lambda; \{\xi_p\}_{N_s}) = \lim_{\epsilon \rightarrow 0} \tilde{P}_{12\dots L}^{(2s)} A_{12\dots L}^{(2s+;\epsilon)}(\lambda; \{w_j^{(2s;\epsilon)}\}_L) \tilde{P}_{12\dots L}^{(2s)}. \quad (4.5)$$

We define the massless monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$ by

$$\begin{aligned} \tilde{T}_{0,12\dots N_s}^{(\ell, 2s+)} &= \tilde{P}_{a_1 a_2 \dots a_\ell}^{(\ell)} \tilde{T}_{a_1, 12\dots N_s}^{(1, 2s+)}(\lambda_{a_1}) \tilde{T}_{a_2, 12\dots N_s}^{(1, 2s+)}(\lambda_{a_1} - \eta) \cdots \\ &\times \cdots \tilde{T}_{a_\ell, 12\dots N_s}^{(1, 2s+)}(\lambda_{a_1} - (\ell - 1)\eta) \tilde{P}_{a_1 a_2 \dots a_\ell}^{(\ell)}. \end{aligned} \quad (4.6)$$

4.2. Integrable spin- s Hamiltonians

We define the massless transfer matrix¹ of type $(\ell, (2s)^{\otimes N_s})$ by

$$\begin{aligned} \tilde{t}_{12\dots N_s}^{(\ell, 2s+)}(\lambda) &= \text{tr}_{V^{(\ell)}} \left(\tilde{T}_{0,12\dots N_s}^{(\ell, 2s+)}(\lambda) \right) = \sum_{n=0}^{\ell} {}_a \langle \ell, n | \tilde{T}_{a_1, 12\dots N_s}^{(1, 2s+)}(\lambda) \times \\ &\times \tilde{T}_{a_2, 12\dots N_s}^{(1, 2s+)}(\lambda - \eta) \cdots \tilde{T}_{a_\ell, 12\dots N_s}^{(1, 2s+)}(\lambda - (\ell - 1)\eta) | \widetilde{|\ell, n\rangle}_a. \end{aligned} \quad (4.7)$$

It follows from the Yang-Baxter equations that the higher-spin transfer matrices commute in the tensor product space $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$, which is derived by applying projection operator $P_{12\dots L}^{(2s)}$ to $V_1^{(1)} \otimes \cdots \otimes V_L^{(1)}$.

The massless spin- s R -matrix $\tilde{R}_{12}^{(2s, 2s+)}(u)$ becomes the permutation operator at $u = 0$: $\tilde{R}_{12}^{(2s, 2s+)}(0) = \Pi_{1,2}$.^{45,50} Therefore, putting inhomogeneous parameters $\xi_p = 0$ for $p = 1, 2, \dots, N_s$, we show that the transfer matrix $\tilde{t}_{12\dots N_s}^{(2s, 2s+)}(\lambda)$ becomes the shift operator at $\lambda = 0$. We derive the massless spin- s XXZ Hamiltonian by the logarithmic derivative of the massless spin- s transfer matrix.

$$\mathcal{H}_{\text{XXZ}}^{(2s)} = \frac{d}{d\lambda} \log \tilde{t}_{12\dots N_s}^{(2s, 2s+)}(\lambda) \Big|_{\lambda=0, \xi_j=0} = \sum_{i=1}^{N_s} \frac{d}{du} \tilde{R}_{i, i+1}^{(2s, 2s)}(u) \Big|_{u=0}. \quad (4.8)$$

5. Spin- $\ell/2$ massless XXZ correlation functions

5.1. Spin- s local operators in terms of global operators

In the massless regime, we can express the Hermitian elementary matrices in terms of global operators as follows.¹ For $m \geq n$ we have

$$\begin{aligned} \tilde{E}_i^{m,n(\ell+)} &= \binom{\ell}{n} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} \tilde{P}_{1\dots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A^{(1+)} + D^{(1+)}) (w_\alpha) \\ &\times \prod_{k=1}^n D^{(1+)}(w_{(i-1)\ell+k}) \prod_{k=n+1}^m B^{(1+)}(w_{(i-1)\ell+k}) \prod_{k=m+1}^{\ell} A^{(1+)}(w_{(i-1)\ell+k}) \\ &\times \prod_{\alpha=i\ell+1}^{\ell N_s} (A^{(1+)} + D^{(1+)}) (w_\alpha) \tilde{P}_{1\dots L}^{(\ell)}. \end{aligned} \quad (5.1)$$

For $m \leq n$ we have

$$\begin{aligned} \tilde{E}_i^{m,n(\ell+)} &= \binom{\ell}{n} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} \tilde{P}_{1\dots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A^{(1+)} + D^{(1+)}) (w_\alpha) \\ &\times \prod_{k=1}^m D^{(1+)}(w_{(i-1)\ell+k}) \prod_{k=m+1}^n C^{(1+)}(w_{(i-1)\ell+k}) \prod_{k=n+1}^{\ell} A^{(1+)}(w_{(i-1)\ell+k}) \\ &\times \prod_{\alpha=i\ell+1}^{\ell N_s} (A^{(1+)} + D^{(1+)}) (w_\alpha) \tilde{P}_{1\dots L}^{(\ell)}. \end{aligned} \quad (5.2)$$

By the quantum inverse-scattering problem (QISP) of Ref. 10 the local spin operators are expressed in terms of global operators and the transfer matrices for the integrable spin- s XXX spin chain. However, it is not clear how one can derive (5.1) and (5.2) by the QISP method even for $q = 1$.

5.2. Symbols for expressing sequences

Let us denote by $(a_j)_m$ a sequence of numbers a_j for $j = 1, 2, \dots, m$, i.e. $(a_j)_m = (a_1, a_2, \dots, a_m)$.

Definition 5.1. We say that a sequence $(b_k)_n$ is a subsequence of $(a_j)_m$ if (i) $n \leq m$, (ii) $b_k \in \{a_1, \dots, a_m\}$ for $k = 1, 2, \dots, n$, (iii) for any pair of integers j and k satisfying $1 \leq j < k \leq n$, there exists a pair of integers $\ell(j)$ and $\ell(k)$ such that $a_j = b_{\ell(j)}$, $a_k = b_{\ell(k)}$ and $\ell(j) < \ell(k)$.

For a pair of sequences $(a_j)_m$ and $(b_k)_n$, we define the product $(a_j)_m \# (b_k)_n$ by a sequence $(c_\ell)_{m+n}$ such that $c_j = a_j$ for $j = 1, 2, \dots, m$ and $c_j = b_j$ for $j = m+1, m+2, \dots, m+n$.

5.3. Conjecture of the spin- s Ground-state solution

Let us now introduce the conjecture that the ground state of the spin- s case $|\psi_g^{(2s+)}\rangle$ is given by $N_s/2$ sets of $2s$ -strings:

$$\lambda_a^{(\alpha)} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)}, \quad \text{for } a = 1, 2, \dots, N_s/2 \text{ and } \alpha = 1, 2, \dots, 2s. \quad (5.3)$$

Here we assume that string deviations $\epsilon_a^{(\alpha)}$ are very small when N_s is very large.[?] In terms of $\lambda_a^{(\alpha)}$, the spin- s ground state in the homogeneous grading is given by¹

$$|\psi_g^{(2s+)}\rangle = \prod_{a=1}^{N_s/2} \prod_{\alpha=1}^{2s} \tilde{B}^{(2s+)}(\lambda_a^{(\alpha)}; \{\xi_p\}_{N_s})|0\rangle. \quad (5.4)$$

We denote by M the number of Bethe roots: $M = 2s N_s/2 = sN_s$.

According to analytic and numerical studies,^{24,37,39,40} we may assume the following properties of string deviations $\epsilon_a^{(\alpha)}$ s. For very large N_s , the deviations are given by $\epsilon_a^{(\alpha)} = i \delta_a^{(\alpha)}$, where i denotes $\sqrt{-1}$ and $\delta_a^{(\alpha)}$ are real. Moreover, $\delta_a^{(\alpha)} - \delta_a^{(\alpha+1)} > 0$ for $\alpha = 1, 2, \dots, 2s-1$, and $|\delta_a^{(\alpha)}| > |\delta_a^{(\alpha+1)}|$ for $\alpha < s$, while $|\delta_a^{(\alpha)}| < |\delta_a^{(\alpha+1)}|$ for $\alpha \geq s$.

In the limit: $N_s \rightarrow \infty$, the density of string centers, $\rho_{\text{tot}}(\mu)$, is given by

$$\rho_{\text{tot}}(\mu) = \frac{1}{N_s} \sum_{p=1}^{N_s} \frac{1}{2\zeta \cosh(\pi(\mu - \xi_p)/\zeta)}. \quad (5.5)$$

For the homogeneous chain where $\xi_p = 0$ for $p = 1, 2, \dots, N_s$, we denote the density of string centers by $\rho(\lambda)$.

$$\rho(\lambda) = \frac{1}{2\zeta \cosh(\pi\lambda/\zeta)}. \quad (5.6)$$

Let us introduce useful notation of the suffix of rapidities. For rapidities $\lambda_a^{(\alpha)} = \lambda_{(a,\alpha)}$ we define integers A by $A = 2s(a-1) + \alpha$ for $a = 1, 2, \dots, N_s/2$ and for $\alpha = 1, 2, \dots, 2s$. We thus denote $\lambda_{(a,\alpha)}$ also by λ_A for $A = 1, 2, \dots, sN_s$, and put $\lambda_{(a,\alpha)}$ in increasing order with respect to $A = 2s(a-1) + \alpha$ such as $\lambda_{(1,1)} = \lambda_1, \lambda_{(1,2)} = \lambda_2, \dots, \lambda_{(N_s/2, 2s)} = \lambda_{sN_s}$. In the ground state, rapidities λ_A for $A = 1, 2, \dots, M$, are expressed by

$$\lambda_{2s(a-1)+\alpha} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)} \quad (1 \leq a \leq N_s/2; 1 \leq \alpha \leq 2s). \quad (5.7)$$

For $A = 2s(a-1) + \alpha$ with $1 \leq \alpha \leq 2s$, integer a is given by $a = [(A-1)/2s] + 1$, and integer α is given by $\alpha = A - 2s[(A-1)/2s]$.

For a real number x we define $[x]$ by the greatest integer less than or equal to x . We define $a(j)$ and $\alpha(j)$ for $j = 1, 2, \dots, M$ as follows.

$$a(j) = [(j-1)/2s] + 1, \quad \alpha(j) = j - 2s[(j-1)/2s]. \quad (5.8)$$

5.4. Correlation functions of the integrable spin- s XXZ model on a long finite chain

We define the correlation function of the integrable spin- $2s$ XXZ spin chain for a given product of $(2s+1) \times (2s+1)$ elementary matrices such as $\tilde{E}_1^{i_1, j_1(2s+)} \dots \tilde{E}_m^{i_m, j_m(2s+)}$ on the spin- s ground state, $|\psi_g^{(2s+)}\rangle$, as follows.

$$F_m^{(2s+)}(\{i_k, j_k\}) = \langle \psi_g^{(2s+)} | \prod_{k=1}^m \tilde{E}_k^{i_k, j_k(2s+)} | \psi_g^{(2s+)} \rangle / \langle \psi_g^{(2s+)} | \psi_g^{(2s+)} \rangle. \quad (5.9)$$

By formulas (5.1) and (5.2) we express the m th product of $(2s+1) \times (2s+1)$ elementary matrices in terms of a $2sm$ th product of 2×2 elementary matrices with entries $\{\epsilon_j, \epsilon'_j\}$ as follows.

$$\prod_{b=1}^m \tilde{E}_b^{i_b, j_b(2s+)} = C(\{i_b, j_b\}) \tilde{P}_{12\dots L}^{(2s)} \cdot \prod_{k=1}^{2sm} e_k^{\epsilon'_k, \epsilon_k} \cdot \tilde{P}_{12\dots L}^{(2s)}. \quad (5.10)$$

By making use of (5.1) and (5.2), $C(\{i_b, j_b\})$ is given by

$$C(\{i_k, j_k\}) = \prod_{b=1}^m \left\{ \binom{2s}{j_b} \begin{bmatrix} 2s \\ i_b \end{bmatrix}_q \begin{bmatrix} 2s \\ j_b \end{bmatrix}_q^{-1} \right\}. \quad (5.11)$$

Here $\epsilon_{2s(b-1)+\beta}$ and $\epsilon'_{2s(b-1)+\beta}$ ($b = 1, \dots, N_s$; $\beta = 1, \dots, 2s$) are given by

$$\epsilon_{2s(b-1)+\beta} = \begin{cases} 1 & (1 \leq \beta \leq j_b) \\ 0 & (j_b < \beta \leq 2s) \end{cases}; \quad \epsilon'_{2s(b-1)+\beta} = \begin{cases} 1 & (1 \leq \beta \leq i_b) \\ 0 & (i_b < \beta \leq 2s) \end{cases}. \quad (5.12)$$

We evaluate the spin- $2s$ XXZ correlation function $F_m^{(2s+)}(\{i_k, j_k\})$ by

$$\begin{aligned} F_m^{(2s+)}(\{i_k, j_k\}) &= C(\{i_k, j_k\}) \langle \psi_g^{(2s+)} | \tilde{P}_{12\dots L}^{(2s)} \times \\ &\quad \times \prod_{j=1}^{2sm} e_j^{\epsilon'_j, \epsilon_j} \cdot \tilde{P}_{12\dots L}^{(2s)} | \psi_g^{(2s+)} \rangle / \langle \psi_g^{(2s+)} | \psi_g^{(2s+)} \rangle \end{aligned} \quad (5.13)$$

Let α^+ be the set of j with $\epsilon_j = 0$, and α^- the set of j with $\epsilon'_j = 1$:

$$\alpha^+ = \{j; \epsilon_j = 0\}, \quad \alpha^- = \{j; \epsilon'_j = 1\}. \quad (5.14)$$

We denote by r and r' the number of elements of the set α^- and α^+ , respectively. Due to charge conservation, we have

$$r + r' = 2sm. \quad (5.15)$$

We denote by j_{\min} and j_{\max} the smallest element and the largest element of α^- , respectively. We also denote by j'_{\min} and j'_{\max} the smallest element and the largest element of α^+ , respectively.

Recall that the ground state $|\psi_g^{(2s+)}\rangle$ has M Bethe roots with $M = sN_s$. Let c_j ($j \in \alpha^-$) and c'_j ($j \in \alpha^+$) be integers such that $1 \leq c_j \leq M$ for $j \in \alpha^-$ and $1 \leq c'_j \leq M+j$ for $j \in \alpha^+$. We define sequence $(b_\ell)_{2sm}$ by

$$(b_1, b_2, \dots, b_{2sm}) = (c'_{j'_{\max}}, \dots, c'_{j'_{\min}}, c_{j_{\min}}, \dots, c_{j_{\max}}). \quad (5.16)$$

Here sequence $(c'_{j'_{\max}}, \dots, c'_{j'_{\min}}, c_{j_{\min}}, \dots, c_{j_{\max}})$ is given by the composite sequence of c'_j s in decreasing order with respect to suffix j , and c_j s in increasing order with respect to suffix j . We introduce the following symbols:

$$\prod_{j \in \alpha^-} \left(\sum_{c_j=1}^M \right) \prod_{j \in \alpha^+} \left(\sum_{c'_j=1}^{M+j} \right) = \sum_{c_{j_{\min}}=1}^M \cdots \sum_{c_{j_{\max}}=1}^M \sum_{c'_{j'_{\min}}=1}^{M+j'_{\min}} \cdots \sum_{c'_{j'_{\max}}=1}^{M+j'_{\max}}. \quad (5.17)$$

Recall that $a(j)$ are defined in (5.8). We define $\beta(j)$ by

$$\beta(j) = j - 2s[(j-1)/2s] \quad (1 \leq j \leq M). \quad (5.18)$$

For $\ell, k = 1, 2, \dots, 2sm$, we define the (ℓ, k) element of $M^{(2sm)}((b_j)_{2sm})$ by

$$\begin{aligned} & \left(M^{(2sm)}((b_j)_{2sm}) \right)_{\ell, k} \\ &= \begin{cases} -\delta_{b_\ell - M, k} & (b_\ell > M) \\ \delta_{\beta(b_\ell), \beta(k)} \cdot \rho(\lambda_{b_\ell} - w_k^{(2s)} + \eta/2) / (N_s \rho_{\text{tot}}(\mu_{a(b_\ell)})) & (b_\ell \leq M) \end{cases} \end{aligned} \quad (5.19)$$

Here, continuous variable μ , which is the argument of density $\rho_{\text{tot}}(\mu)$, is evaluated at $\mu_{a(b_\ell)}$, one of the “string centers” μ_a of $2s$ -strings (5.7).

We can rigorously derive a concise expression of correlation functions of the spin- s XXZ spin chain in the massless region: $0 \leq \zeta < \pi/2s$ for a large finite chain. Introducing $\varphi(\lambda) = \sinh \lambda$ we have

$$\begin{aligned} F_m^{(2s+)}(\{i_b, j_b\}) &= C(\{i_k, j_k\}) \prod_{j \in \alpha^-} \left(\sum_{c_j=1}^M \right) \prod_{j \in \alpha^+} \left(\sum_{c'_j=1}^{M+j} \right) \det M^{(2sm)}((b_\ell)_{2sm}) \\ &\times (-1)^r \frac{\prod_{j \in \alpha^-} \left(\prod_{k=1}^{j-1} \varphi(\lambda_{c_j} - w_k^{(2s)} + \eta) \prod_{k=j+1}^{2sm} \varphi(\lambda_{c_j} - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(\lambda_{b_\ell} - \lambda_{b_k} + \eta)} \\ &\times \frac{\prod_{j \in \alpha^+} \left(\prod_{k=1}^{j-1} \varphi(\lambda_{c'_j} - w_k^{(2s)} - \eta) \prod_{k=j+1}^{2sm} \varphi(\lambda_{c'_j} - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(w_k^{(2s)} - w_\ell^{(2s)})} + O(1/N_s). \end{aligned} \quad (5.20)$$

We remark that we derive (5.20) sending ϵ to zero. Before taking the limit, inhomogeneous parameters w_j s are generic due to small parameter ϵ , and the sums over variables c_j in (5.20) are restricted up to M for all j .

5.5. Multiple-integral representations of spin- s XXZ correlation function for arbitrary matrix elements

In the thermodynamic limit: $N_s \rightarrow \infty$, rapidities λ_{b_ℓ} with b_ℓ defined in (5.16), correspond to integral variables λ_ℓ for $\ell = 1, 2, \dots, 2sm$. For $1 \leq b_\ell \leq M$ they are given by the Bethe roots of $2s$ -strings (5.7), while for $b_\ell > M$ they are given by complete $2s$ -strings $w_k^{(2s)}$ defined by (4.1).

We define $\alpha(\lambda_j)$ by $\alpha(\lambda_j) = \gamma$ for an integer γ with $1 \leq \gamma \leq 2s$, if λ_j is related to integral variable μ_j by $\lambda_j = \mu_j - (\gamma - 1/2)\eta$, or if λ_j takes a value close to $w_k^{(2s)}$ with $\beta(k) = \gamma$, where $w_k^{(2s)}$ are part of complete $2s$ -strings (4.1). Here, variables μ_j correspond to “string centers” of variables λ_j .

We define the (j, k) element of matrix $S = S((\lambda_j)_{2sm}; (w_j^{(2s)})_{2sm})$ by

$$S_{j,k} = \rho(\lambda_j - w_k^{(2s)} + \eta/2) \delta(\alpha(\lambda_j), \beta(k)), \quad \text{for } j, k = 1, 2, \dots, 2sm. \quad (5.21)$$

Here $\delta(\alpha, \beta)$ denotes the Kronecker delta, and we recall (5.18) for $\beta(k)$.

Let Γ_j be a small contour rotating counterclockwise around $\lambda = w_j^{(2s)}$. Since $\det S$ has simple poles at $\lambda = w_j^{(2s)}$ with residue $1/2\pi i$, we have

$$\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \det S((\lambda_k)_{2sm}) d\lambda_1 = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \det S((\lambda_k)_{2sm}) d\lambda_1 - \oint_{\Gamma_1} \det S((\lambda_k)_{2sm}) d\lambda_1. \quad (5.22)$$

For sets α^- and α^+ with relation (5.16), we define integral variables $\tilde{\lambda}_j$ for $j \in \alpha^-$ and $\tilde{\lambda}'_j$ for $j \in \alpha^+$, respectively, by the following:

$$(\tilde{\lambda}'_{j'_{max}}, \dots, \tilde{\lambda}'_{j'_{min}}, \tilde{\lambda}_{j_{min}}, \tilde{\lambda}_{j_{max}}) = (\lambda_1, \dots, \lambda_{2sm}). \quad (5.23)$$

Thus, from expression (5.20) of the correlation function in terms of a finite sum, we derive the multiple-integral representation as follows.

$$\begin{aligned} F_m^{(2s+)}(\{i_k, j_k\}) &= C(\{i_b, j_b\}) \times \\ &\times \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \dots + \int_{-\infty-i\tilde{\zeta}_s+i\epsilon}^{\infty-i\tilde{\zeta}_s+i\epsilon} \right) d\lambda_1 \dots \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \dots + \int_{-\infty-i\tilde{\zeta}_s+i\epsilon}^{\infty-i\tilde{\zeta}_s+i\epsilon} \right) d\lambda_{r'} \\ &\times \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \dots + \int_{-\infty-i\tilde{\zeta}_s-i\epsilon}^{\infty-i\tilde{\zeta}_s-i\epsilon} \right) d\lambda_{\tilde{r}} \dots \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \dots + \int_{-\infty-i\tilde{\zeta}_s-i\epsilon}^{\infty-i\tilde{\zeta}_s-i\epsilon} \right) d\lambda_{2sm} \\ &\times Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) \det S(\lambda_1, \dots, \lambda_{2sm}). \end{aligned} \quad (5.24)$$

Here $\tilde{\zeta}_s = (2s-1)\zeta$, $\tilde{r} = r' + 1$, and $Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm})$ is given by

$$\begin{aligned} & Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) \\ &= (-1)^{r'} \frac{\prod_{j \in \alpha^-} \left(\prod_{k=1}^{j-1} \varphi(\tilde{\lambda}_j - w_k^{(2s)} + \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_j - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(\lambda_\ell - \lambda_k + \eta + \epsilon_{\ell,k})} \\ & \times \frac{\prod_{j \in \alpha^+} \left(\prod_{k=1}^{j-1} \varphi(\tilde{\lambda}'_j - w_k^{(2s)} - \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}'_j - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(w_k^{(2s)} - w_\ell^{(2s)})}. \end{aligned} \quad (5.25)$$

In the denominator we set $\epsilon_{k,\ell} = i\epsilon$ for $\text{Im}(\lambda_k - \lambda_\ell) > 0$ and $\epsilon_{k,\ell} = -i\epsilon$ for $\text{Im}(\lambda_k - \lambda_\ell) < 0$, where ϵ is infinitesimally small: $|\epsilon| \ll 1$. Here, $\text{Im}(a+ib) = b$ for real numbers a and b . Here, for α^\pm , we recall (5.14).

We evaluate $\alpha(\lambda_j)$ in (5.24), replacing paths $(-\infty - i(\gamma-1)\zeta \pm i\epsilon, \infty - i(\gamma-1)\zeta \pm i\epsilon)$ by $(-\infty - i(\gamma-1/2)\zeta, \infty - i(\gamma-1/2)\zeta)$ for $\gamma = 1, 2, \dots, 2s$, respectively. The integrals over λ_j for $j \geq \tilde{r}$ do not change when $\epsilon \rightarrow \zeta/2$.

Thus, correlation functions (5.9) are expressed in the form of a single term of multiple integrals (5.24).

We can derive the symmetric expression for the multiple-integral representations of the spin- s correlation function $F_m^{(2s+)}(\{i_k, j_k\})$ as follows.¹

$$\begin{aligned} & F_m^{(2s+)}(\{i_k, j_k\}) = C(\{i_b, j_b\}) \times \\ & \times \frac{1}{\prod_{1 \leq \alpha < \beta \leq 2s} \sinh^m(\beta - \alpha)\eta} \prod_{1 \leq k < l \leq m} \frac{\sinh^{2s}(\pi(\xi_k - \xi_l)/\zeta)}{\prod_{j=1}^{2s} \prod_{r=1}^{2s} \sinh(\xi_k - \xi_l + (r-j)\eta)} \\ & \times \sum_{\sigma \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}} (\text{sgn } \sigma) \prod_{j=1}^{r'} \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \dots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\mu_{\sigma j} \\ & \times \prod_{j=r'+1}^{2sm} \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \dots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\mu_{\sigma j} \\ & \times Q(\{\epsilon_j, \epsilon'_j\}; \lambda_{\sigma 1}, \dots, \lambda_{\sigma(2sm)}) \left(\prod_{j=1}^{2sm} \frac{\prod_{b=1}^m \prod_{\beta=1}^{2s-1} \sinh(\lambda_j - \xi_b + \beta\eta)}{\prod_{b=1}^m \cosh(\pi(\mu_j - \xi_b)/\zeta)} \right) \\ & \times \frac{i^{2sm^2}}{(2i\zeta)^{2sm}} \prod_{\gamma=1}^{2s} \prod_{1 \leq b < a \leq m} \sinh(\pi(\mu_{2s(a-1)+\gamma} - \mu_{2s(b-1)+\gamma})/\zeta). \end{aligned} \quad (5.26)$$

Here λ_j are given by $\lambda_j = \mu_j - (\beta(j) - 1/2)\eta$ for $j = 1, \dots, 2sm$.

It is straightforward to take the homogeneous limit: $\xi_k \rightarrow 0$. Here $(\text{sgn } \sigma)$ denotes the sign of permutation $\sigma \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$.

6. Derivation of finite-sum expression of spin- s XXZ correlation functions with arbitrary entries

6.1. Fundamental commutation relations

We now discuss briefly the derivation of (5.20), which expresses the spin- s XXZ correlation functions with arbitrary entries in terms of the product of finite sums over the Bethe roots.

Let Σ_N be the set of integers $1, 2, \dots, N$, i.e. $\Sigma_N = \{1, 2, \dots, N\}$. Recall definition (5.14) of α^\pm and that of integers c_j and c'_j . For a given set of c_j, c'_j , we introduce \mathbf{A}_j and \mathbf{A}'_j by

$$\begin{aligned}\mathbf{A}_j &= \{b; 1 \leq b \leq M + 2sm, b \neq c_k, c'_k \text{ for } k < j\}, \\ \mathbf{A}'_j &= \{b; 1 \leq b \leq M + 2sm, b \neq c_k \text{ for } k \leq j, b \neq c'_k \text{ for } k < j\}.\end{aligned}\quad (6.1)$$

We define sets α_j^\pm and $c(\alpha_j^\pm)$ as follows.

$$\alpha_j^- = \{k; k < j, k \in \alpha^-\}, \quad \alpha_j^+ = \{k; k < j, k \in \alpha^+\}, \quad (6.2)$$

$$c(\alpha_j^-) = \{c_k; k \in \alpha_j^-\}, \quad c(\alpha_j^+) = \{c'_k; k \in \alpha_j^+\}. \quad (6.3)$$

We have

$$\mathbf{A}_j = \Sigma_{M+2sm} \setminus (c(\alpha_j^-) \cup c(\alpha_j^+)), \quad \mathbf{A}'_j = \Sigma_{M+2sm} \setminus (c(\alpha_{j+1}^-) \cup c(\alpha_{j+1}^+)).$$

Let us denote by t the number of c_j ($j \in \alpha^-$) and c'_j ($j \in \alpha^+$) such that $c_j, c'_j \leq M$, for a given set of c_j and c'_j . We express (5.17) as follows.

$$\sum_{t=r}^{2sm} \sum_{\{c_j, c'_j\}_t} = \prod_{j \in \alpha^-} \left(\sum_{c_j=1}^M \right) \prod_{j \in \alpha^+} \left(\sum_{c'_j=1}^{M+j} \right). \quad (6.4)$$

Here the sum over $\{c_j, c'_j\}_t$ denotes the sums over c_j and c'_j such that the number of $c'_j \leq M$ is fixed by $t - r$.

Suppose that λ_α for $\alpha = 1, 2, \dots, M$ give a set of solutions of the Bethe ansatz equations in the spin-1/2 case with $w_j = w_j^{(2s; \epsilon)}$ for $j = 1, 2, \dots, L$.¹ Here w_j are inhomogeneous parameters. We set rapidities λ_{M+j} by

$$\lambda_{M+j} = w_j, \quad j = 1, 2, \dots, 2sm. \quad (6.5)$$

We can show the fundamental commutation relations as follows.¹¹

$$\begin{aligned}& \langle 0 | \left(\prod_{\alpha=1}^M C(\lambda_\alpha) \right) T_{\epsilon_1, \epsilon'_1}(\lambda_{M+1}) \cdots T_{\epsilon_{2sm}, \epsilon'_{2sm}}(\lambda_{M+2sm}) \\ &= \sum_{t=r}^{2sm} \sum_{\{c_j, c'_j\}_t} G_{\{c_j, c'_j\}}(\lambda_1, \dots, \lambda_{M+2sm}) \langle 0 | \prod_{k \in \mathbf{A}_{2sm+1}(\{c_j, c'_j\})} C(\lambda_k),\end{aligned}$$

where $d(\lambda; \{w_k^{(2s;\epsilon)}\}_L)$ and $G_{\{c_j, c'_j\}}((\lambda_\alpha)_{M+2sm})$ are given by

$$\begin{aligned}
 d(\lambda; \{w_k^{(2s;\epsilon)}\}_L) &= \prod_{k=1}^L b(\lambda - w_k^{(2s;\epsilon)}), \\
 G_{\{c_j, c'_j\}}(\lambda_1, \dots, \lambda_{M+2sm}) &= \prod_{j \in \alpha^+} \left(\frac{\prod_{b=1; b \in \mathbf{A}'_j}^{M+j-1} \varphi(\lambda_b - \lambda_{c'_j} + \eta)}{\prod_{b=1; b \in \mathbf{A}_{j+1}}^{M+j} \varphi(\lambda_b - \lambda_{c'_j})} \right) \\
 &\quad \times \prod_{j \in \alpha^-} \left(d(\lambda_{c_j}; \{w_k^{(2s;\epsilon)}\}_L) \frac{\prod_{b=1; b \in \mathbf{A}_j}^{M+j-1} \varphi(\lambda_{c_j} - \lambda_b + \eta)}{\prod_{b=1; b \in \mathbf{A}'_j}^{M+j} \varphi(\lambda_{c_j} - \lambda_b)} \right). \tag{6.6}
 \end{aligned}$$

6.2. Finite-sum expression of correlation functions for a finite chain

We introduce disjoint subsets of α^+ , α_J^+ and α_K^+ , as follows.

$$\alpha_J^+ = \{j; j \in \alpha^+, 1 \leq c'_j \leq M\}, \quad \alpha_K^+ = \{j; j \in \alpha^+, c'_j > M\}. \tag{6.7}$$

We define sets $c(\alpha^-)$, $c(\alpha_J^+)$ and $c(\alpha_K^+)$ as follows.

$$c(\alpha^-) = \{c_k; k \in \alpha^-\}, \quad c(\alpha_J^+) = \{c_k; k \in \alpha_J^+\}, \quad c(\alpha_K^+) = \{c_k; k \in \alpha_K^+\}.$$

We define a sequence $(\tilde{b}_k)_t$ by a subsequence of $(b_k)_{2sm}$ such that $\tilde{b}_k \leq M$ for $k = 1, 2, \dots, t$. We denote sequence $(b_k)_{2sm}$ and $(\tilde{b}_k)_t$ as sets by \mathbf{b} and $\tilde{\mathbf{b}}_t$, respectively, i.e. $\mathbf{b} = \{b_1, b_2, \dots, b_{2sm}\}$ and $\tilde{\mathbf{b}}_t = \{\tilde{b}_1, \dots, \tilde{b}_t\}$. Here we note $\tilde{\mathbf{b}}_t = c(\alpha^-) \cup c(\alpha_J^+)$. We define sequence $(b'_k)_{2sm-t}$ by a subsequence of $(b_k)_{2sm}$ such that $b'_k > M$ for $k = 1, 2, \dots, 2sm-t$. We denote it as a set by \mathbf{b}'_{2sm-t} . Here we note $\mathbf{b}'_{2sm-t} = c(\alpha_K^+)$.

We define sets Z and K by $Z = \Sigma_M \setminus \tilde{\mathbf{b}}_t$ and $K = \Sigma_{2sm} \setminus \mathbf{b}'_{2sm-t}$, respectively. We define a sequence $(z(\alpha))_{M-t}$ by putting the elements of Z in increasing order: $z(1) < z(2) < \dots < z(M-t)$ where $Z = \{z(i); i = 1, 2, \dots, M-t\}$, and a sequence $(\kappa_j)_t$ by putting the elements of K in increasing order: $\kappa_1 < \kappa_2 < \dots < \kappa_t$ where $K = \{\kappa_j; j = 1, 2, \dots, t\}$.

We derive the spin- s correlation functions from those of the spin-1/2 case sending ϵ to zero:

$$F_m^{(2s+)}(\{i_b, j_b\}; (w_j^{(2s;+)}))_L = C(\{i_k, j_k\}) \lim_{\epsilon \rightarrow 0} F_{2sm}^{(1+)}(\{\epsilon_j, \epsilon'_j\}; (w_j^{(2s;\epsilon)}))_L. \tag{6.8}$$

Applying (6.6) to (5.9) (or (5.13)) we have

$$\begin{aligned}
F_{2sm}^{(1+)}(\{\epsilon_j, \epsilon'_j\}; (w_j^{(2s; \epsilon)})_L) &= \sum_{t=r}^{2sm} \sum_{\{c_j, c'_j\}_t} G_{\{c_j, c'_j\}}(\lambda_1, \dots, \lambda_{M+2sm}) \\
&\times \phi_{2sm}(\{\lambda_\alpha\}_M) \frac{\langle 0 | \prod_{\alpha=1}^{M-t} C(\lambda_{z(\alpha)}) \prod_{\gamma=1}^t C(w_{\kappa_\gamma}) \prod_{\beta=1}^{M-t} B(\lambda_{z(\beta)}) \prod_{\gamma=1}^t B(\lambda_{b_\gamma}^-) | 0 \rangle}{\langle 0 | \prod_{\alpha=1}^{M-t} C(\lambda_{z(\alpha)}) \prod_{\gamma=1}^t C(w_{b_\gamma}^-) \prod_{\beta=1}^{M-t} B(\lambda_{z(\beta)}) \prod_{\gamma=1}^t B(\lambda_{b_\gamma}^-) | 0 \rangle} \\
&= \sum_{t=r}^{2sm} \sum_{\{c_j, c'_j\}_t} \prod_{\alpha=1}^M \prod_{j=1}^{2sm} \frac{\varphi(\lambda_\alpha - w_j)}{\varphi(\lambda_\alpha - w_j + \eta)} \prod_{j \in \alpha^-} \left(\frac{\prod_{b=1, b \in A_j}^{M+j-1} \varphi(\lambda_{c_j} - \lambda_b + \eta)}{\prod_{b=1, b \in A'_j}^{M+j} \varphi(\lambda_{c_j} - \lambda_b)} \right) \\
&\times \prod_{j \in \alpha^+} \left(\frac{\prod_{b=1, b \in A'_j}^{M+j-1} \varphi(\lambda_b - \lambda_{c'_j} + \eta)}{\prod_{b=1, b \in A_{j+1}}^{M+j} \varphi(\lambda_b - \lambda_{c'_j})} \right) \prod_{1 \leq k < \ell \leq t} \frac{\varphi(\lambda_{b_k}^- - \lambda_{b_\ell}^-)}{\varphi(w_{\kappa_k} - w_{\kappa_\ell})} \\
&\times \prod_{\alpha=1}^{M-t} \prod_{\ell=1}^t \frac{\varphi(\lambda_{z(\alpha)} - \lambda_{b_\ell}^-)}{\varphi(\lambda_{z(\alpha)} - w_{\kappa_\ell})} \prod_{\alpha=1}^M \prod_{\ell=1}^t \frac{\varphi(\lambda_\alpha - w_{\kappa_\ell} + \eta)}{\varphi(\lambda_\alpha - \lambda_{b_\ell}^- + \eta)} \\
&\times \det \left((\Phi')^{-1} \left(\lambda_{z(\alpha)} \right)_{M-t} \# (\lambda_{b_\ell}^-)_t \right) \times \\
&\times \Psi' \left((\lambda_{z(\alpha)} \right)_{M-t} \# (w_{\kappa_\ell})_t, (\lambda_{z(\alpha)} \right)_{M-t} \# (\lambda_{b_\ell}^-)_t \right). \quad (6.9)
\end{aligned}$$

Here, $\phi_m(\{\lambda_\alpha\}) = \prod_{j=1}^m \prod_{\alpha=1}^M b(\lambda_\alpha - w_j)$, and matrix elements $(\Psi')_{ab}$ for $a, b = 1, 2, \dots, M$ are given by

$$\begin{aligned}
&\left(\Psi' \left((\lambda_{z(\alpha)} \right)_{M-t} \# (\lambda_{b_\ell}^-)_t, (\lambda_{z(\alpha)} \right)_{M-t} \# (w_{\kappa_\ell})_t; (w_k)_L \right)_{a,b} \\
&= \begin{cases} \Phi'_{a,b} \left((\lambda_{z(\alpha)} \right)_{M-t} \# (\lambda_{b_\ell}^-)_t \right) & \text{for } b \leq M-t \\ \frac{\varphi(\eta)}{\varphi(\lambda_{z(a)} - w_{\kappa_k}) \varphi(\lambda_{z(a)} - w_{\kappa_k} + \eta)} & \text{for } b = k + M-t \ (1 \leq k \leq t) \end{cases} \quad (6.10)
\end{aligned}$$

The matrix elements of the Gaudin matrix are given as follows.

$$\begin{aligned}
\Phi'_{a,b} \left((\lambda_{z(\alpha)} \right)_{M-t} \# (\lambda_{b_\ell}^-)_t; (w_k)_L \right) &= \Phi'_{z(a), z(b)}((\lambda_\alpha)_M; (w_k)_L) \\
&= \frac{\varphi(2\eta)}{\varphi(\lambda_a - \lambda_b + \eta) \varphi(\lambda_a - \lambda_b - \eta)} + \delta_{a,b} \left(\sum_{p=1}^L \frac{\varphi(\eta)}{\varphi(\lambda_a - w_p) \varphi(\lambda_a - w_p + \eta)} \right. \\
&\quad \left. - \sum_{\gamma=1}^M \frac{\varphi(2\eta)}{\varphi(\lambda_a - \lambda_\gamma + \eta) \varphi(\lambda_a - \lambda_\gamma - \eta)} \right). \quad (6.11)
\end{aligned}$$

For any positive integer N_s we can rigorously calculate (6.9) as follows.⁵²

Proposition 6.1.

$$\begin{aligned}
 F_{2sm}^{(1+)}(\{\epsilon_j, \epsilon'_j\}; (w_j^{(2s; \epsilon)})_L) &= \sum_{t=r}^{2sm} \sum_{\{c_j, c'_j\}} \left(\prod_{j, k \in \alpha_K^+, c'_j < c'_k, j < k} (-1) \right) \\
 &\times (-1)^{2sm-t} \prod_{j \in \alpha_K^+} \left(\prod_{\ell \in \alpha_j^+; \ell > j} (-1) \cdot \prod_{\kappa \in K; \kappa + M < c'_j} (-1) \right) \\
 &\times \det(\Phi')^{-1} \Psi'((\lambda_{z(\alpha)})_{M-t} \# (\xi_{\kappa_\ell})_t, (\lambda_{z(\alpha)})_{M-t} \# (\lambda_{b_\ell}^-)_t) \times \\
 &\times (-1)^{r'} \frac{\prod_{j \in \alpha^-} \left(\prod_{k=1}^{j-1} \varphi(\lambda_{c_j} - w_k^{(2s; \epsilon)} + \eta) \prod_{k=j+1}^{2sm} \varphi(\lambda_{c_j} - w_k^{(2s; \epsilon)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(\lambda_{b_\ell} - \lambda_{b_k} + \eta)} \\
 &\times \frac{\prod_{j \in \alpha^+} \left(\prod_{k=1}^{j-1} \varphi(\lambda_{c'_j} - w_k^{(2s; \epsilon)} - \eta) \prod_{k=j+1}^{2sm} \varphi(\lambda_{c'_j} - w_k^{(2s; \epsilon)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(w_k^{(2s; \epsilon)} - w_\ell^{(2s; \epsilon)})}.
 \end{aligned} \tag{6.12}$$

We define matrix elements (j, k) of $\phi_M^{(2sm)}((b_\ell)_{2sm})$ ($1 \leq j \leq 2sm$):⁵²

$$\begin{aligned}
 \text{If } b_j > M, \quad & \left(\phi_M^{(2sm)}((b_\ell)_{2sm}) \right)_{j, k} = -\delta_{b_j - M, k} \quad \text{for } k = 1, 2, \dots, 2sm, \\
 \text{if } b_j \leq M, \quad & \text{there is an integer } i \text{ such that } b_j = \tilde{b}_i \\
 & \left(\phi_M^{(2sm)}((b_\ell)_{2sm}) \right)_{j, \kappa_k} = (\Phi')^{-1} \Psi'((\lambda_{z(\alpha)})_{M-t} \# (\lambda_{b_\ell}^-)_t, \\
 & \quad (\lambda_{z(\alpha)})_{M-t} \# (\xi_{\kappa_\ell})_t)_{i+M-t, k+M-t}, \quad \text{for } k = 1, 2, \dots, t, \\
 \text{and } & \phi_M^{(2sm)}((b_\ell)_{2sm})_{j, b'_k} = 0 \quad \text{for } k = 1, 2, \dots, 2sm - t.
 \end{aligned} \tag{6.13}$$

We can show the following proposition.⁵²

Proposition 6.2.

$$\begin{aligned}
 & \det \left((\Phi')^{-1} \Psi'((\lambda_{z(\alpha)})_{M-t} \# (w_{\kappa_\ell})_t, (\lambda_{z(\alpha)})_{M-t} \# (\lambda_{b_\ell}^-)_t) \right) \\
 &= \det \phi_M^{(2sm)}((b_\ell)_{2sm}) (-1)^{2sm-t} \left(\prod_{j, k \in \alpha_K^+, c'_j < c'_k, j < k} (-1) \right) \\
 & \times \prod_{j \in \alpha_K^+} \left(\prod_{\ell \in \alpha_j^+; \ell > j} (-1) \cdot \prod_{\kappa \in K; \kappa + M < c'_j} (-1) \right).
 \end{aligned} \tag{6.14}$$

When N_s is large enough, solving the integral equations for $\phi_M^{(2sm)}((b_\ell)_{2sm})$, we can show

$$\det \phi_M^{(2sm)}((b_\ell)_{2sm}) = \det M^{(2sm)}((b_\ell)_{2sm}) + O(1/N_s). \quad (6.15)$$

We thus obtain the finite-size spin- s XXZ correlation functions with arbitrary entries (5.20).

Acknowledgments

One of the authors (T.D.) would also like to thank K. Motegi for helpful collaboration on the spin- s R -matrices in Ref. 50, which are closely related to the present study. Furthermore, the authors would like to thank S. Miyashita for encouragement and keen interest in this work. The authors are grateful to the organizers of the workshop “Infinite Analysis 09– New Trends in Quantum Integrable Systems –”, July 27–31, 2009, Kyoto University, Japan. This work is partially supported by Grant-in-Aid for Scientific Research (C) No. 20540365.

\bibliographystyle{ws-procs9x6}
 \% \bibliography{Deguchi}

References

1. T. Deguchi and C. Matsui, Correlation functions of the integrable higher-spin XXX and XXZ spin chains through the fusion method, Nucl. Phys. B. **831** [FS] (2010) 359–407.
2. M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, Correlation functions of the XXZ model for $\Delta < -1$, Phys. Lett. **A 168** (1992) 256–263.
3. M. Jimbo and T. Miwa, Algebraic Analysis of Solvable Lattice Models (AMS, Providence, RI, 1995).
4. M. Jimbo and T. Miwa, Quantum KZ equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime, J. Phys. A: Math. Gen. **29** (1996) 2923–2958.
5. T. Miwa and Y. Takeyama, Determinant Formula for the Solutions of the Quantum Knizhnik-Zamolodchikov Equation with $|q| = 1$, Contemporary Mathematics **248** (1999) 377–393.
6. N.A. Slavnov, Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz, Theor. Math. Phys. **79** (1989) 502–508.
7. V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993)
8. J.M. Maillet and J. Sanchez de Santos, Drinfel’d twists and algebraic Bethe ansatz, ed. M. Semenov-Tian-Shansky, Amer. Math. Soc. Transl. **201** Ser. 2, (Providence, R.I.: Amer. Math. Soc., 2000) pp. 137–178.

9. N. Kitanine, J.M. Maillet and V. Terras, Form factors of the XXZ Heisenberg spin-1/2 finite chain, Nucl. Phys. B **554** [FS] (1999) 647–678.
10. J.M. Maillet and V. Terras, On the quantum inverse scattering problem, Nucl. Phys. B **575** [FS] (2000) 627–644.
11. N. Kitanine, J.M. Maillet and V. Terras, Correlation functions of the XXZ Heisenberg spin-1/2 chain in a magnetic field, Nucl. Phys. B **567** [FS] (2000) 554–582.
12. F. Göhmann, A. Klümper and A. Seel, Integral representations for correlation functions of the XXZ chain at finite temperature, J. Phys. A: Math. Gen. **37** (2004) 7625–7651.
13. J. Damerau, F. Göhmann, N. Hasenclever and A. Klümper, Density matrices for finite segments of Heisenberg chains of arbitrary length, J. Phys. A: Math. Theor., **40**, 4439 (2007).
14. M. Jimbo, T. Miwa and F. Smirnov, Hidden Grassmann structure in the XXZ model III: Introducing the Matsubara direction, J. Phys. A: Math. Theor. **42**, 304018 (2009).
15. N. Kitanine, K.K. Kozłowski, J.M. Maillet, N.A. Slavnov, V. Terras, Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions, J. Stat. Mech. (2009) P04003.
16. N. Kitanine, Correlation functions of the higher spin XXX chains, J. Phys. A: Math. Gen. **34**(2001) 8151–8169.
17. O. A. Castro-Alvaredo and J. M. Maillet, Form factors of integrable Heisenberg (higher) spin chains, J. Phys. A: Math. Theor. **40** (2007) 7451–7471.
18. T. Deguchi and C. Matsui, Form factors of integrable higher-spin XXZ chains and the affine quantum-group symmetry, Nucl. Phys. B. **814** [FS] (2009) 405–438.
19. P.P. Kulish, N. Yu. Reshetikhin and E.K. Sklyanin, Yang-Baxter equation and representation theory: I, Lett. Math. Phys. **5** (1981) 393–403.
20. H. M. Babujian, Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model, Nucl. Phys. B **215** [FS7] (1983) 317–336.
21. H. M. Babujian and A. M. Tsvelick, Heisenberg magnet with an arbitrary spin and anisotropic chiral field, Nucl. Phys. B **265** [FS15] (1986) 24–44.
22. A.B. Zamolodchikov and V.A. Fateev, A model factorized S -matrix and an integrable spin-1 Heisenberg chain, Sov. J. Nucl. Phys. **32** (1980) 298–303.
23. K. Sogo, Y. Akutsu and T. Abe, New Factorized S -Matrix and Its Application to Exactly Solvable q -State Model. I; II, Prog. Theor. Phys. **70** (1983) 730–738; 739–746.
24. A. N. Kirillov and N. Yu. Reshetikhin, Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. the ground state and the excitation spectrum, J. Phys. A: Math. Gen. **20** (1987) 1565–1585.
25. T. Deguchi, M. Wadati and Y. Akutsu, Exactly Solvable Models and New Link Polynomials. V. Yang-Baxter Operator and Braid-Monoid Algebra, J. Phys. Soc. Jpn. **57** (1988) 1905–1923.
26. L.A. Takhtajan, The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins, Phys. Lett. A **87** (1982) 479–482.

27. K. Sogo, Ground state and low-lying excitations in the Heisenberg XXZ chain of arbitrary spin S , Phys. Lett. A **104** (1984) 51–54.
28. H. Johannesson, Central charge for the integrable higher-spin XXZ model, J. Phys. A: Math. Gen. **21** (1988) L611–L614.
29. H. Johannesson, Universality classes of critical antiferromagnets, J. Phys. A: Math. Gen. **21** (1988) L1157–L1162.
30. F.C. Alcaraz and M.J. Martins, Conformal invariance and critical exponents of the Takhtajan-Babujian models, J. Phys. A: Math. Gen. **21** (1988) 4397–4413.
31. I. Affleck, D. Gepner, H.J. Schultz and T. Ziman, Critical behavior of spin- s Heisenberg antiferromagnetic chains: analytic and numerical results, J. Phys. A: Math. Gen. **22** (1989) 511–529.
32. B.-D. Dörfel, Finite-size corrections for spin- S Heisenberg chains and conformal properties, J. Phys. A: Math. Gen. **22** (1989) L657–L662.
33. L.V. Avdeev, The lowest excitations in the spin- s XXX magnet and conformal invariance, J. Phys. A: Math. Gen. **23** (1990) L485–L492.
34. F.C. Alcaraz and M.J. Martins, Conformal invariance and the operator content of the XXZ model with arbitrary spin, J. Phys. A: Math. Gen. **22** (1989) 1829–1858.
35. H. Frahm, N. -C. Yu and M. Fowler, The integrable XXZ Heisenberg model with arbitrary spin: construction of the Hamiltonian, the ground-state configuration and conformal properties, Nucl. Phys. B **336** (1990) 396–434.
36. H. Frahm and N. -C. Yu, Finite-size effects in the XXZ Heisenberg model with arbitrary spin, J. Phys. A: Math. Gen. **23** (1990) 2115–2132.
37. H. J. de Vega and F. Woynarovich, Solution of the Bethe ansatz equations with complex roots for finite size: the spin $S \geq 1$ isotropic and anisotropic chains, J. Phys. A: Math. Gen. **23** (1990) 1613–1626.
38. A. Klümper and M. T. Batchelor, An analytic treatment of finite-size corrections in the spin-1 antiferromagnetic XXZ chain, J. Phys. A: Math. Gen. **23** (1990) L189–L195.
39. A. Klümper, M. T. Batchelor and P. A. Pearce, Central charge of the 6- and 19-vertex models with twisted boundary conditions, J. Phys. A: Math. Gen. **24** (1991) 3111–3133.
40. J. Suzuki, Spinons in magnetic chains of arbitrary spins at finite temperatures, J. Phys. A: Math. Gen. **32** (1999) 2341–2359.
41. M. Idzumi, Calculation of Correlation Functions of the Spin-1 XXZ Model by Vertex Operators, Thesis, University of Tokyo, Feb. 1993.
42. M. Idzumi, Level two irreducible representations of $U_q(\widehat{sl_2})$, vertex operators, and their correlations, Int. J. Mod. Phys. A **9** (1994) 4449–4484.
43. A. H. Bougourzi and R. A. Weston, N -point correlation functions of the spin 1 XXZ model, Nucl. Phys. B **417** (1994) 439–462.
44. H. Konno, Free-field representation of the quantum affine algebra $U_q(\widehat{sl_2})$ and form factors in the higher-spin XXZ model, Nucl. Phys. B **432** [FS] (1994) 457–486.
45. M. Jimbo, A q -Difference Analogue of $U(g)$ and the Yang-Baxter Equation, Lett. Math. Phys. **10** (1985) 63–69.

46. M. Jimbo, A q -analogue of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation, Lett. Math. Phys. **11** (1986) 247–252.
47. V.G. Drinfel'd, Quantum groups, Proc. ICM Berkeley 1986, pp. 798–820.
48. R. J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic Press, 1982).
49. Y. Akutsu and M. Wadati, Exactly Solvable Models and New Link polynomials. I. N -State Vertex Models, J. Phys. Soc. Jpn. **56** (1987) 3039–3051.
50. T. Deguchi and K. Motegi, in preparation.
51. M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models*, (Cambridge University Press, Cambridge, 1999).
52. T. Deguchi, in preparation.